Equation 3-6 presumes that the uncertainty in each factor of the product \( x \cdot z \) is random and independent of the other. In the product \( x \cdot z \), the measured value of \( x \) could be high sometimes and the measured value of \( z \) could be low sometimes. In the majority of cases, the uncertainty in the product \( x \cdot z \) is not as great as the uncertainty in \( x^2 \).

**TEST YOURSELF** You can calculate the time it will take for an object to fall from the top of a building to the ground if you know the height of the building. If the height has an uncertainty of 1.0%, what is the uncertainty in time? (Answer: 0.5%)

If \( y \) is the base 10 logarithm of \( x \), then the absolute uncertainty in \( y (e_y) \) is proportional to the relative uncertainty in \( x \), which is \( e_x/x \):

\[
y = \log x \quad \Rightarrow \quad e_y = \frac{e_x}{x} \quad (3-8)
\]

You should not work with percent relative uncertainty \( [100 \times (e_y/x)] \) in calculations with logs and antilogs because one side of Equation 3-8 has relative uncertainty and the other has absolute uncertainty.

The **natural logarithm** (ln) of \( x \) is the number \( y \), whose value is such that \( e = e^y \), where \( e (= 2.718 28...) \) is called the base of the natural logarithm. The absolute uncertainty in \( y \) is equal to the relative uncertainty in \( x \).

\[
y = \ln x \quad \Rightarrow \quad e_y = \frac{e_x}{x} \quad (3-9)
\]

Now consider \( y = \text{antilog} x \), which is the same as saying \( y = 10^x \). In this case, the relative uncertainty in \( y \) is proportional to the absolute uncertainty in \( x \).

\[
y = 10^x \quad \Rightarrow \quad \frac{e_y}{y} = (\ln 10) e_x \approx 2.302 6 \ e_x \quad (3-10)
\]

If \( y = e^x \), the relative uncertainty in \( y \) equals the absolute uncertainty in \( x \).

\[
y = e^x \quad \Rightarrow \quad \frac{e_y}{y} = e_x \quad (3-11)
\]

Table 3-1 summarizes rules for propagation of uncertainty. You need not memorize the rules for exponents, logs, and antilogs, but you should be able to use them.

![Table 3-1](image)

**TABLE 3-1** Summary of rules for propagation of uncertainty

<table>
<thead>
<tr>
<th>Function</th>
<th>Uncertainty</th>
<th>Function</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = x_1 + x_2 )</td>
<td>( e_y = \sqrt{e_{x_1}^2 + e_{x_2}^2} )</td>
<td>( y = x^a )</td>
<td>( %e_y = a %e_x )</td>
</tr>
<tr>
<td>( y = x_1 - x_2 )</td>
<td>( e_y = \sqrt{e_{x_1}^2 + e_{x_2}^2} )</td>
<td>( y = \log x )</td>
<td>( e_y = \frac{e_x}{\ln 10 \ x} \approx 0.434 29 \ e_x )</td>
</tr>
<tr>
<td>( y = x_1 \cdot x_2 )</td>
<td>( %e_y = \sqrt{%e_{x_1}^2 + %e_{x_2}^2} )</td>
<td>( y = \ln x )</td>
<td>( e_y = \frac{e_x}{x} )</td>
</tr>
<tr>
<td>( y = \frac{x_1}{x_2} )</td>
<td>( %e_y = \sqrt{%e_{x_1}^2 + %e_{x_2}^2} )</td>
<td>( y = 10^x )</td>
<td>( \frac{e_y}{y} = (\ln 10) \ e_x \approx 2.302 6 \ e_x )</td>
</tr>
<tr>
<td>( y = e^x )</td>
<td>( e_y = e_x )</td>
<td>( y = e^x )</td>
<td>( e_y = e_x )</td>
</tr>
</tbody>
</table>

* a. \( x \) represents a variable and \( a \) represents a constant that has no uncertainty.
* b. \( e_x/x \) is the relative error in \( x \) and \( \%e_x \) is \( 100 \times e_x/x \).

**EXAMPLE** Uncertainty in \( \text{H}^+ \) Concentration

Consider the function \( \text{pH} = -\log [\text{H}^+] \), where \( [\text{H}^+] \) is the molarity of \( \text{H}^+ \). For \( \text{pH} = 5.21 \pm 0.03 \), find \([\text{H}^+]\) and its uncertainty.