1.2 Vectors

Vector is a new concept associated to a space and designed to study the space. It is proved to be useful in solving many different problems about the space. One may think of a vector as a transportation assignment in the space: the assignment says how far and in which direction the transportation should be made. The distance of the transportation is called the **magnitude** of the vector. The transportation assignment can be applied to any point of the Euclidean space and two assignments are said to be the same if their directions and magnitudes coincide.

The set of all vectors associated to the space is so important that it deserves a name on its own — it is called a vector space. The main difference between the original Euclidean space and the associated vector space is that there is a natural operation available in the vector space — any two assignments can be composed to produce another assignment. This operation is usually called **addition** of vectors. As we will see later, this operation satisfies several laws and therefore imposes certain restriction on the vector space. It may sound a bit paradoxical but these laws make the vector space much easier to study (as opposed to studying the original Euclidean space). The reason for that is the laws are very familiar to us from studying numbers and thus provide us with additional effective tools when discovering, stating, and using properties of the vector space.

1.2.1 Basic properties of vectors

Suppose that $P(p_1, p_2)$ and $Q(q_1, q_2)$ are two points in Euclidean coordinate plane. The transportation assignment from P to Q can be described by the changes in coordinates needed for the transportation: the x-coordinate changes by $q_1 - p_1$ and the y-coordinate changes by $q_2 - p_2$. Thus the vector $\overrightarrow{PQ}$ can be described by the pair of numbers $\langle q_1 - p_1, q_2 - p_2 \rangle$. We will use angular brackets for vector notations as opposed to usual brackets used for coordinates of points.

Vectors can be added component-wise: if $\overrightarrow{v} = \langle v_1, v_2 \rangle$ and $\overrightarrow{w} = \langle w_1, w_2 \rangle$ are two vectors, then

$$\overrightarrow{v} + \overrightarrow{w} = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle = \langle v_1 + w_1, v_2 + w_2 \rangle$$

A vector can be multiplied by any real number λ as follows:

$$\lambda \overrightarrow{v} = \langle \lambda v_1, \lambda v_2 \rangle$$

The resulting vector $\lambda \overrightarrow{v}$ is collinear with the vector $\overrightarrow{v}$. Moreover, it has the same direction as $\overrightarrow{v}$ if λ is positive and the opposite direction if λ is negative.

1.2.2 Collinearity using proportionality of vectors

The basic idea we use in this section is that two vectors are collinear if and only if they are proportional. This idea allows us to check geometric linearity by very simple algebra.

Vectors can be used to determine if three points on a plane belong to one line (i.e. if the points are **collinear**). Three points P, Q, and R are collinear if and only if the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are proportional. That means $\overrightarrow{PQ} = \lambda\overrightarrow{PR}$ for some number λ. The number λ may be positive or negative depending on whether the directions of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are the same or opposite.

Problem. 3 points on line

Determine whether the points lie on a straight line: $P(1,2)$, $Q(-1,5)$, $R(2,1)$.

Generalization. Determine whether three points of a plane lie on a straight line. □

Strategy. We consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and see if the equation $\overrightarrow{PQ} = \lambda\overrightarrow{PR}$ can be satisfied for some number λ. We conclude the points are collinear if such a number λ exists and not collinear otherwise. □

Computation. Since $\overrightarrow{PQ} = \langle -2, 3\rangle$ and $\overrightarrow{PR} = \langle 1, -1\rangle$, we have $\overrightarrow{PQ} = \lambda\overrightarrow{PR}$ equivalent to the system of two equations: $\begin{cases} -2 = \lambda\,(1) \\ 3 = \lambda\,(-1) \end{cases}$

So, the first equation has only one solution $\lambda = -2$ while the second has only one solution $\lambda = -3$. Thus there is no solution to the system and the points are not collinear. □

1.2.3 Orthogonality of vectors

Given two vectors $\overrightarrow{u} = \langle u_1, u_2\rangle$ and $\overrightarrow{v} = \langle v_1, v_2\rangle$, we can determine if these vectors are orthogonal by checking that the Pythagoras Theorem holds for the triangle formed by the given vectors. This triangle has sides of length $|\overrightarrow{u}| = \sqrt{{u_1}^2 + {u_2}^2}$, $|\overrightarrow{v}| = \sqrt{{v_1}^2 + {v_2}^2}$, and $|\overrightarrow{u} - \overrightarrow{v}| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$. Thus we have to check if the sum $|\overrightarrow{u}|^2 + |\overrightarrow{v}|^2$ is equal to $|\overrightarrow{u} - \overrightarrow{v}|^2$. This amounts to checking that the quantity $|\overrightarrow{u}|^2 + |\overrightarrow{v}|^2 - |\overrightarrow{u} - \overrightarrow{v}|^2$ is equal to zero. $|\overrightarrow{u}|^2 + |\overrightarrow{v}|^2 - |\overrightarrow{u} - \overrightarrow{v}|^2 = {u_1}^2 + {u_2}^2 + {v_1}^2 + {v_2}^2 - \left((u_1 - v_1)^2 + (u_2 - v_2)^2\right) = 2\,(u_1v_1 + u_2v_2)$

This quantity is equal to zero if and only if $u_1v_1 + u_2v_2$ equals zero. Thus two vectors $\overrightarrow{u} = \langle u_1, u_2\rangle$ and $\overrightarrow{v} = \langle v_1, v_2\rangle$ are orthogonal if and only if $u_1v_1 + u_2v_2 = 0$.

We have just discovered a useful expression $u_1v_1 + u_2v_2$ associated to any pair of vectors $\overrightarrow{u} = \langle u_1, u_2\rangle$ and $\overrightarrow{v} = \langle v_1, v_2\rangle$. This expression provides a number called **dot product**. The standard notation for the dot product of two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ is $\overrightarrow{v}\cdot\overrightarrow{w}$. Note that the dot product is commutative, i.e. for any vectors $\overrightarrow{u}$, $\overrightarrow{v}$, we have $\overrightarrow{v}\cdot\overrightarrow{u} = \overrightarrow{u}\cdot\overrightarrow{v}$. Why is it called "product"? For the single reason that it is **distributive** with respect to addition of vectors, i.e. for any vectors $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$, the following equality holds: $(\overrightarrow{u} + \overrightarrow{v})\cdot\overrightarrow{w} = \overrightarrow{u}\cdot\overrightarrow{w} + \overrightarrow{v}\cdot\overrightarrow{w}$. This product appears to be useful for the following two reasons: it has an important geometric meaning and it has a simple algebraic expression which makes it easy to compute.

Let us notice at this point that the dot product of a vector with itself is equal to

$$\overrightarrow{u} \cdot \overrightarrow{u} = u_1^2 + u_2^2 = |\overrightarrow{u}|^2$$

the square of the magnitude of the vector!

Problem. Orthogonality of vectors in plane

Determine whether the given vectors are orthogonal: $\overrightarrow{u} = \langle 1, 2\rangle$, $\overrightarrow{v} = \langle 2, -1\rangle$.

Generalization. Determine whether the given vectors are orthogonal: $\overrightarrow{u} = \langle u_1, u_2\rangle$, $\overrightarrow{v} = \langle v_1, v_2\rangle$. □

Strategy. The vectors are orthogonal if and only if $u_1v_1 + u_2v_2 = 0$. □

Computation. $u_1v_1 + u_2v_2 = 1 \cdot 2 + 2 \cdot (-1) = 0$. So, the vectors are orthogonal. □

Problem. 3 points right triangle

Given three points $P(1, 2)$, $Q(-1, 5)$, $R(2, 1)$ on a plane. Determine whether they form a right triangle.

Generalization. Given three points on a plane. Determine whether they form a right triangle. □

Strategy. We consider three vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{RQ}$ and verify if any two of them are perpendicular. Two vectors $\overrightarrow{u} = \langle u_1, u_2\rangle$ and $\overrightarrow{v} = \langle v_1, v_2\rangle$ are perpendicular if and only if $u_1v_1 + u_2v_2 = 0$. □

Computation. $\overrightarrow{PQ} = \langle -2, 3\rangle$, $\overrightarrow{PR} = \langle 1, -1\rangle$ $\overrightarrow{RQ} = \langle -3, 4\rangle$.

The expressions $u_1v_1 + u_2v_2$ for these vectors are:
$\overrightarrow{PQ}$ and $\overrightarrow{PR}$: $-2 \cdot 1 + 3 \cdot (-1) = -5$,
$\overrightarrow{PQ}$ and $\overrightarrow{RQ}$: $-2 \cdot (-3) + 3 \cdot 4 = 18$,
$\overrightarrow{RQ}$ and $\overrightarrow{PR}$: $(-3) \cdot 1 + 4 \cdot (-1) = -7$.

Since none of these expressions is zero, no two sides of the triangle are perpendicular, thus the triangle is not right. □